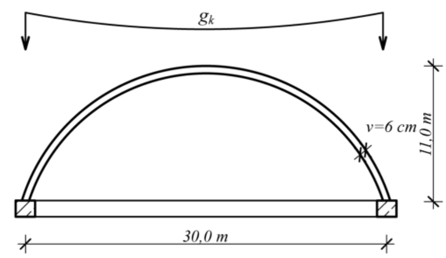
T3/1 – Internal forces of spherical dome loaded by self-weight



1. Data

Material:

specific weight of reinforced concrete: ρ=25 kN/m^3

Geometry:

L =30,0 m (span)

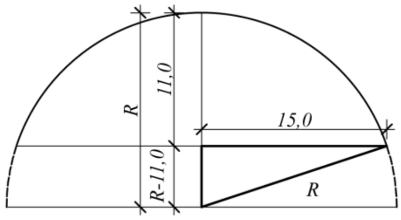
f =11,0 m (height)

v = 6 cm (thickness)

From the geometrical data, the radius of the sphere can be calculated using *Pythagoras' theorem*:

$$(L/2)^2 + (R - f)^2 = R^2$$

$$R = 15{,}72 \text{ m}$$



2. Loads

Characteristic value of the self-load is

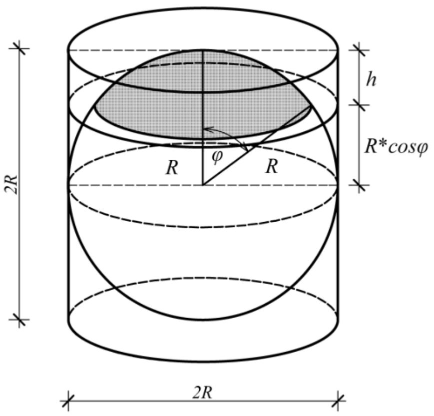
$g_k = \rho v = 25 \cdot 0{,}06 = 1{,}5$ kN/m^2.

Design value of the self-load is

$g_d = g_k \gamma_g = 1{,}35 \cdot 1{,}5 = 2{,}25$ kN/m^2,

where $\gamma_g = 1{,}35$ is the safety factor for self-load (note that for live loads, the safety factor is $\gamma_g = 1{,}5$).

To obtain the self-weight (i.e. the resultant of the self-load), we need to calculate the surface area of a spherical cap. We are interested about the internal forces in each points of the dome, hence we derive a general formula for the self-load depending on the φ angle. A single value of φ can be assigned to each point of the dome and we take into consideration the surface area of the spherical cap above the point.



Archimedes' theorem: The surface area of the cylinder which can be drawn around the sphere is equal to the surface of the spherical cap of equal height.

The area of the cylinder:

$$A = 2R\pi \cdot h.$$

The height of the cylinder in terms of φ:

$$h = R - R\cdot\cos\varphi = R(1-\cos\varphi).$$

Hence,

$$\boxed{A = 2\pi R^2(1-\cos\varphi).}$$

The self-weight (concentrated load) of the spherical cap in terms of φ is

$$\boxed{G_d = A \cdot g_d = 2\pi R^2 g_d (1-\cos\varphi).}$$

<u>Additional notes:</u> *in case of a **flat dome**, the loads can be approximated by a uniformly distributed load over the base circle of the dome (p_{Ed}). In that case the radius of the base circle in a point with an angle φ is:*

$$r_{base} = R\sin(\varphi),$$

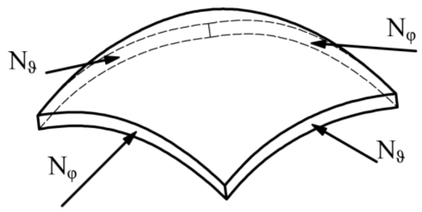
and the resultant would be:

$$\boxed{P_{Ed} = A_{base}\, p_{Ed} = r_{base}^2\, \pi p_{Ed} = \left(R\sin(\varphi)\right)^2 \pi p_{Ed}.}$$

(If you can use the flat dome assumption, it is mentioned in the description of the exercise.)

3. Internal forces (= *membrane forces = normal forces [kN/m])*
(The membrane forces (resultant stresses) are defined as the integration of stresses along the thickness: $N = \int_{-h}^{h} \sigma dz$*)*

We cut out an infinitesimal element of the shell to examine the stress distribution in the dome. It can be defined by two angles, a co-latitude φ and a second co-ordinate ϑ. The stress resultants N_φ and N_ϑ acting in the meridional and hoop directions (along the parallel circles) respectively maintain equilibrium of the shell against its self-weight.

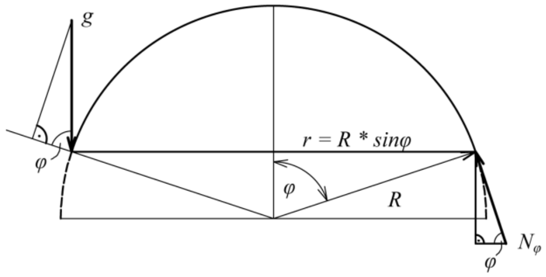


Equations:

- Static equilibrium equation (from *Barlow's formula*):

1. $\boxed{\frac{N_\varphi}{R}+\frac{N_\vartheta}{R}=-g_d\cos\varphi}$

where N_φ: meridian force *(a longitudinal circle)*
N_ϑ: force acting along the parallel circles (in the hoop direction)



- Vertical equilibrium:

2. $\boxed{G_d+KN_\varphi\sin\varphi=0}$

- Perimeter of the base circle:

3. $K=2\pi R\sin\varphi$; $(R\sin\varphi=r)$

- The self-weight of the spherical cap (derived in step 2):

4. $G_d=Ag_d=2\pi R^2 g_d(1-\cos\varphi)$

From equations (2) and (3):

$$G_d+2\pi RN_\varphi\sin^2\varphi=0 \qquad \textit{/identity: } \sin^2\varphi=1-\cos^2\varphi=(1-\cos\varphi)(1+\cos\varphi)/$$

substituted into equation No.4:

$$2\pi R^2 g_d(1-\cos\varphi)=-2\pi RN_\varphi\cdot(1-\cos\varphi)(1+\cos\varphi)$$

after simplification:

$$Rg_d=-N_\varphi(1+\cos\varphi) \qquad \rightarrow \boxed{N_\varphi=\frac{-R\cdot g_d}{1+\cos\varphi}} \ \text{[kN/m]}$$

determination of N_ϑ from static equilibrium equation *(*1)

$$\frac{-R\cdot g_d}{(1+\cos\varphi)\cdot R}+\frac{N_\vartheta}{R}=-g_d\cdot\cos\varphi \rightarrow \boxed{N_\vartheta=g_d R\left(\frac{+1}{1+\cos\varphi}-\cos\varphi\right) \text{[kN/m]}}$$

Now we can assign a φ and ϑ to each points of the surface and the internal forces can be calculated using the derived formulas.

<u>*Additional notes:*</u> *in case of a* ***flat dome****, equation 2 and 4 are slightly different (instead of* G_d *we have* P_{Ed}*), resulting in different formulas for* N_φ *and* N_ϑ*! For the sake of completeness, we also give their formula here:*

$$\boxed{N_\varphi=\frac{-p_{Ed}R}{2}, N_\vartheta=\frac{p_{Ed}R}{2}(1-2\cos^2\varphi)}$$

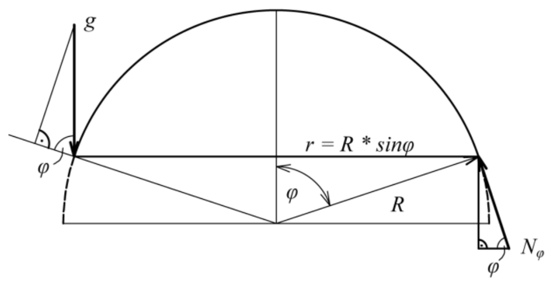
4. Internal force diagrams (normal force diagrams):

To calculate the internal forces in the spherical dome given in Step 1, we need to calculate the φ value for the edge of the dome (at the bottom). Taking R = 15,72 m and r = 15 m:

$$\varphi_k = \arcsin\left(\frac{15}{15{,}72}\right) = 72{,}6° .$$

At the top point of the dome: φ=0°.

In case of a hemisphere, φ=90° at the bottom points.



Now we use the formulas derived in Step 3.

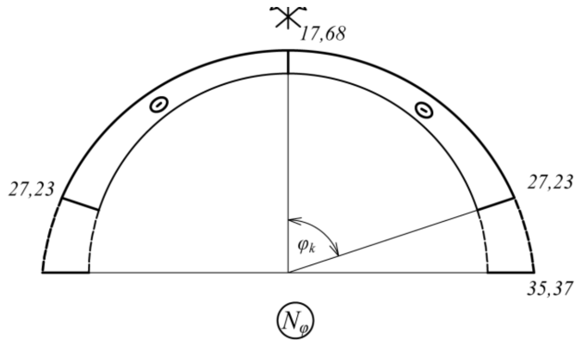
Meridional direction

$$N_\varphi^{\varphi=0°} = \frac{-15{,}72 \cdot 2{,}25}{1+\cos 0°} = -17{,}68 \ \mathbf{kN/m}$$

$$N_\varphi^{\varphi=72{,}6°} = \frac{-15{,}72 \cdot 2{,}25}{1+\cos 72{,}6°} = -27{,}23 \ \mathbf{kN/m}$$

if it was a hemisphere:

$$N_\varphi^{\varphi=90°} = \frac{-15{,}72 \cdot 2{,}25}{1+\cos 90°} = -35{,}37 \ \mathbf{kN/m}$$



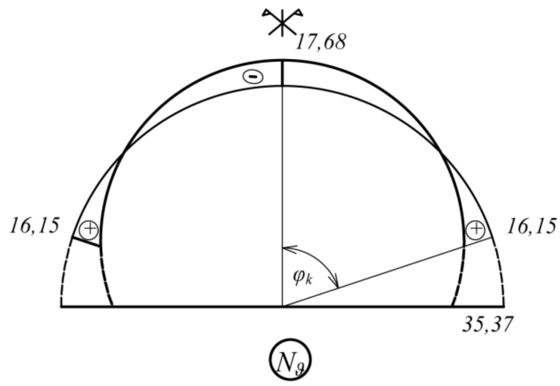
Parallel direction

$$N_\vartheta^{\varphi=0°} = 2{,}25 \cdot 15{,}72\left(\frac{1}{1+\cos 0°} - \cos 0°\right) = -17{,}68 \ \mathbf{kN/m}$$

$$N_\vartheta^{\varphi=72{,}6°} = 2{,}25 \cdot 15{,}72\left(\frac{1}{1+\cos 72{,}6°} - \cos 72{,}6°\right) = +16{,}65 \ \mathbf{kN/m}$$

if it was a hemisphere:

$$N_\vartheta^{\varphi=90°} = 2{,}25 \cdot 15{,}72\left(\frac{1}{1+\cos 90°} - \cos 90°\right) = +35{,}37 \ \mathbf{kN/m}$$



<u>At the bottom of the spherical cap tensile force occurs in the parallel direction!</u>

Where does the sign of the N_ϑ diagram change? (N_ϑ=0, φ=?) Using the formula derived in Step 3 for N_ϑ:

$$\frac{1}{1+\cos\varphi} - \cos\varphi = 0$$

$$0 = \cos^2\varphi + \cos\varphi - 1 \rightarrow \boxed{\varphi = 51{,}82°}$$

Imporant observation: for incomplete domes with internal angle smaller 51,82° (flat domes), only compression occurs in the parallel direction. If the internal angle is larger, tensile forces develop around the base.

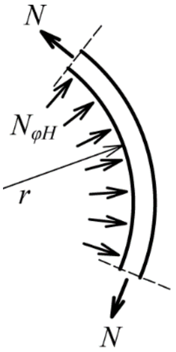
5. Determine the normal force acting in the edge beam!

We have already calculated, the φ_k angle for the edge. The horizontal component of $N_\varphi^{\varphi=\varphi_k}$ acts on the edge beam (for help have a look at the figure on the top of the page):

$$N_{\varphi_H}=N_\varphi^{\varphi=\varphi_k}\cdot\cos\varphi_k=27{,}23\cdot\cos72{,}6=\underline{8{,}14\text{ kN/m}}$$

From the static equilibrium equation, the normal force in the edge beam can be calculated:

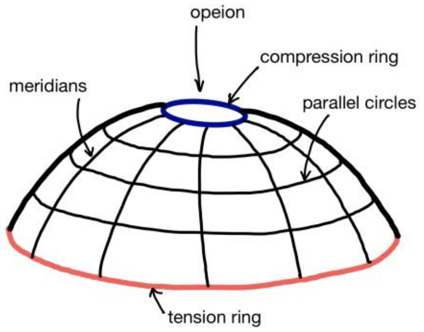
$N_{edge}=N_{\varphi_H}\cdot r=8{,}14\cdot15=\boxed{+\,122{,}15\text{ kN}}$ → *there is tensile force in the edge beam!*



Problem of edge-disturbance

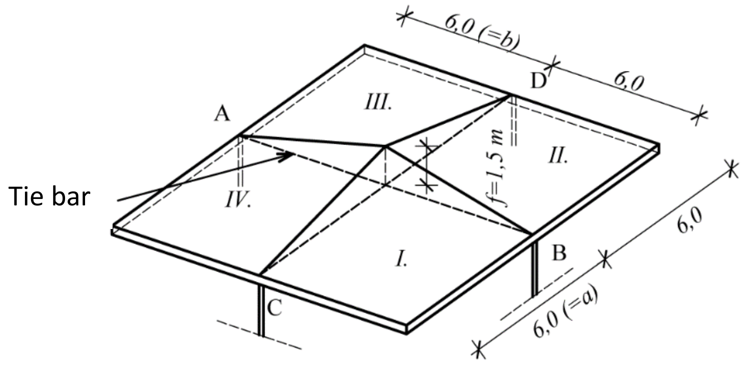
For flat domes (φ<51,82°) only compressive forces occur along the parallel circles. Otherwise, the membrane forces generate tensile force in the (rigid) edge beam (as we have just calculated). The resulting deformations (the length of the parallel circles would decrease, while that of the edge beam would increase) are incompatible hence moment will develop to compensate.

Additional notes: *The bottom edge is called 'tension ring'. If there is an opening on the top of the shell (opeion), it is supported by another edge beam called 'compression ring'. There is always tension in the bottom edge and compression in the top edge independent of the height of the structure.*



T3/2 – Examination of force distribution in an umbrella roof (hyperbolic paraboloid, hypar, quadrilateral)

1. Data



self-weight: g_k=1,5 kN/m^2 (skew)
snow: s_k=1,5 kN/m^2 (vertical)

2. Loads

The safety factors are $\gamma_q = 1{,}5$ (for live load) and $\gamma_g = 1{,}35$ (for self-load). Hence, the design values of the loads:

$g_d = g_k\gamma_g$=1,35·1,5=2,025 kN/m^2

$s_d = s_k\gamma_q$=1,5·1,5=2,25 kN/m^2

The total load of the structure is:

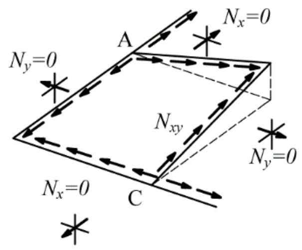
$$P_{Ed} = g_d + s_d = 4{,}28\ kN/m^2$$

a) Determine the normal force acting on the columns!

The resultant of the total load is $P_{Ed} \cdot A$, where A is the area of the roof. This resultant force is distributed between four columns, therefore the force acting on one column is:

$$N_{\mathrm{Ed}}^{\mathrm{column}} = 4{,}28 \cdot \frac{12 \cdot 12}{4} = \boxed{153{,}9\ \mathrm{kN}}\ \text{(vertical support reaction)}$$

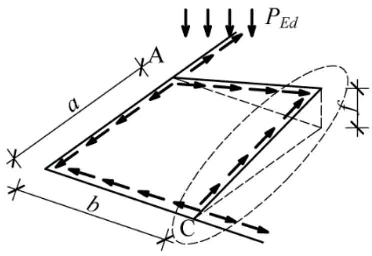
b) Determine the normal force acting on the tie bar!

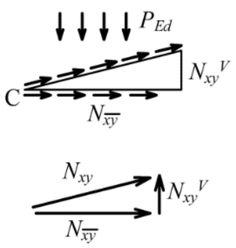


First, we have to analyze the force distribution in the shell.

The vertical load is carried by the vertical components of the N_{xy} membrane shear forces acting along the edges. *(Note, that this is also a stress resultant-type force we discussed in case of the dome! [kN/m])*

At the edges N_x=0, N_Y=0 (normal forces)! (since the local curvature=0).





Semi-rigid edges are required as support for a membrane stress-state. The shear forces acting on the shell result normal forces in the edge beams.

<u>Forces in the semi-rigid edges:</u>

One skew edge equilibrates: $P_{Ed} \cdot a \cdot b/2$ [kN] force.

(Note, that there are two skew edges!)

Therefore, the vertical component of the force in a semi-rigid edge is:

$$N_{\mathrm{xy}}^{\mathrm{V}} = \frac{P_{\mathrm{Ed}} \cdot a \cdot b}{2} \cdot \frac{1}{a} = \frac{4{,}28 \cdot 6{,}0}{2} = 12{,}84\ \text{kN/m}$$ (we divided it by 'a' to obtain a distributed force).

Form the vertical component, we can calculate the horizontal component of the force:

$$\overline{N_{\mathrm{xy}}} = N_{\mathrm{xy}}^{\mathrm{V}} \cdot \frac{a}{f} = \frac{P_{\mathrm{Ed}} \cdot a \cdot b}{2 \cdot a} \cdot \frac{a}{f} = \frac{P_{\mathrm{Ed}} \cdot a \cdot b}{2 \cdot f} = 51{,}36\ \text{kN/m}$$ *(in case of a reasonably flat shell* $\overline{N_{\mathrm{xy}}} \approx N_{\mathrm{xy}}$.

Now the force in the skew edge can be calculated from its components:

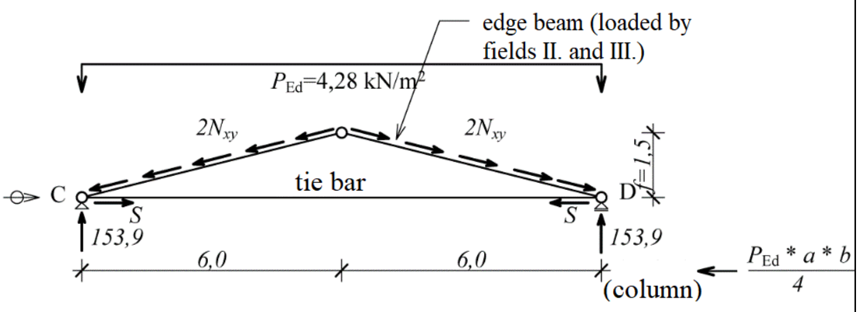
$$N_{xy} = \sqrt{12{,}84^2 + 51{,}36^2} = 52{,}9 \text{ kN/m}.$$

Now it is clear how the load P_{Ed} is distributed between the edges. From now on, instead of P_{Ed}, we use N_{xy} in the calculations.

Forces in the tie bars:

One tie bar is connected to two semi-rigid edges. From the equilibrium equations:

$$S = 2 \cdot \overline{N_{xy}} \cdot a = 2 \cdot 51{,}36 \cdot 6{,}0$$
$$= 616{,}32\ kN$$



Vertical support reactions:

Can be calculated as in part **a)** or as the resultant of the vertical components of shear forces:

$$N_{Ed}^{column} = 2 \cdot a \cdot N_{xy}^{V} = 154{,}0 \text{ kN}$$

c) Calculate the support reactions and the normal forces in the tie bars in case of asymmetrical loading: only fields I and II are loaded by snow, fields III and IV carry only their self-weight!

On fields I and II, the total load has been already calculated, and the horizontal components of the distributed forces acting in the semi-rigid edges were:

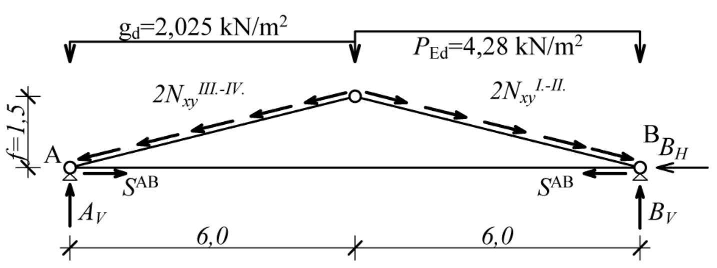
$$\overline{N_{xy}^{\text{I-II}}} = 51{,}36 \text{ kN/m}.$$

The same steps can be done for fields III and IV, but the total load is only g_d:

$$\overline{N_{xy}^{\text{III-IV}}} = \frac{g_d \cdot a \cdot b}{2 \cdot f} = \frac{2{,}025 \cdot 6 \cdot 6}{2 \cdot 1{,}5} = 24{,}3 \text{ kN/m}$$

We work through the same steps as in part b) and use N_{xy}^{I-II} and N_{xy}^{III-IV} to understand how P_{Ed} and g_d are transferred to the supports.

Asymmetrical loading A-B direction



Direction A-B:

$$S^{AB} = 2\,\overline{N_{xy}^{III-IV}}\,b = 2 \cdot 24{,}3 \cdot 6{,}0 = 291{,}6\ kN$$

$$B_H = 2\overline{N_{xy}^{I-II}}\,b = 2 \cdot 51{,}36 \cdot 6{,}0 - S^{AB} = 324{,}7\ kN$$

$$A_V = \frac{1{,}5}{6} \cdot 2 \cdot 24{,}3 \cdot 6 = \underline{73{,}44 \text{ kN}}$$

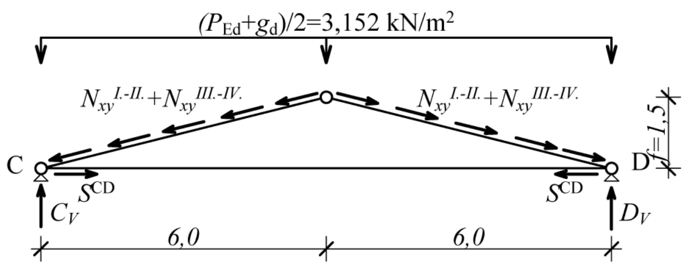
$$B_V = \frac{1{,}5}{6} \cdot 2 \cdot 51{,}36 \cdot 6 = \underline{154{,}1 \text{ kN}}$$

Direction C-D:

$$S^{CD} = (51{,}36 + 24{,}3) \cdot 6 = 453{,}96 \text{ kN}$$

$$C_V = D_V = \frac{1{,}5}{6} \cdot (51{,}36 + 24{,}3) \cdot 6 = 113{,}49 \text{ kN}$$

Asymmetrical loading C-D direction



d) Design the cross sections of the tie bars and the semi rigid edges by calculation!

We design based on the requirement, that the acting for should be less, than the resistance of the cross-section: $N_{Ed} \leq N_{Rd}$.

Materials:

steel: f_{yd}=355 N/mm^2 (S355),

concrete: f_{cd}=16,6 N/mm^2

- **Design of the tie bars*:*** solid steel rod under pure tension, d = ?

Maximal acting force in the tie bars:

N_{Ed}=616,32 kN=616,32·1000 N

Resistance of the cross section:

$N_{Rd}=\frac{A \cdot f_{yd}}{\gamma_{Mo}}$, where $\gamma_{Mo} = 1$ is the safety factor for steel and A is the area of the cross-section.

$$\rightarrow A_{required}=\frac{N_{Rd}}{f_{yd}}=\frac{616{,}36 \cdot 1000}{355}=1736{,}22 \text{ mm}^2 \rightarrow r=\sqrt{\frac{A_{required}}{\pi}}=23{,}5 \text{ mm} \rightarrow \boldsymbol{d = 50 \text{ mm}}$$

(r and d denote the radius and the diameter of the rod, respectively.)

- **Design of the semi-rigid edges*:*** in-situ reinforced concrete, b=250 mm (given data); h=?

The maximal acting force in a semi-rigid edge (resultant of the distributed force):

$$N_{Ed} = N_{xy} \cdot 2 \cdot 6{,}0 = 52{,}9 \cdot 2 \cdot 6{,}0 = 634{,}8 \text{ kN}$$

Estimation: in the case of pure compression (no eccentricity) it can be assumed that the compressive force is carried by the concrete cross section, while the steel rebars take up the moments due to buckling.

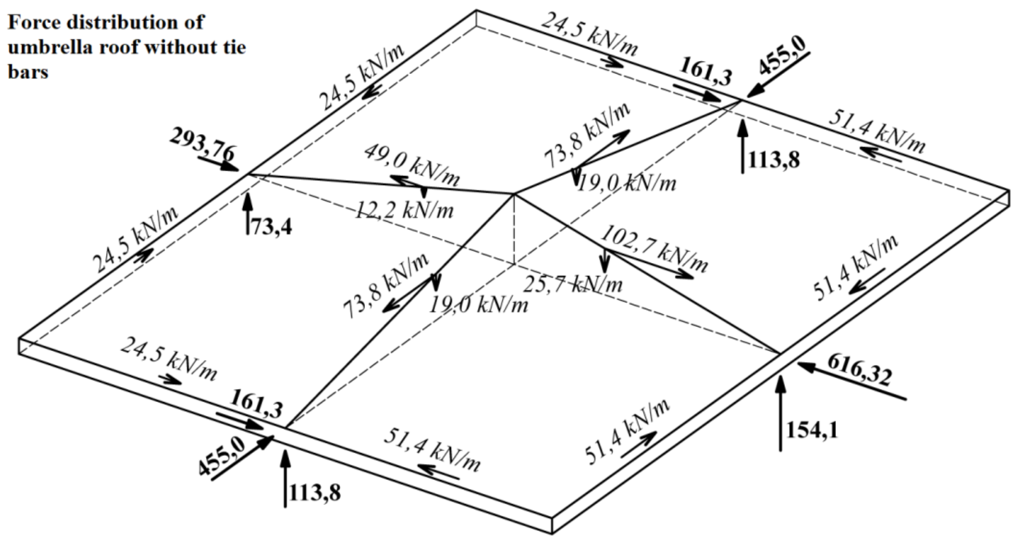
We require, that

$$N_{Rd} = b \cdot h \cdot f_{cd} \geq N_{Ed}.$$

Therefore:

N_{Rd}=250·h·16,6 ≥634,8·10^3 → h≥153,0 mm. Let it be $\boxed{h\text{=25 cm}}$!

e) Analyze the case without tie bars!



Without tie bars, the supports carry the horizontal forces.